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ON EQUATIONS WITH UNIVERSAL INVARIANCE

D. R. Grigore¹

Dept. Theor. Phys., Inst. Atomic Phys.,
Bucharest-Măgurele, P. O. Box MG 6, ROMÂNIA

Abstract

A general discussion of equations with universal invariance for a scalar field is provided in the framework of Lagrangian theory of first-order systems.

¹e-mail: grigore@roifa.bitnet, grigore@ifa.ro

1 Introduction

Recently there has been some interest in the study of the partial differential equations having the so-called *universal invariance* [1]-[4]. For a field with N components this means that if Φ^A ($A = 1, 2, \dots, N$) is a solution of the field equations, then $F \circ \Phi$ is also a solution of the same equation for any diffeomorphism $F \in \text{Diff}(\mathbb{R}^N)$. One usually supposes that such an equation follows from a variational principle i.e. is of the Lagrangian type.

The principle of universal invariance seems to produce many interesting equations of physical relevance. So, it will be desirable to have a program of classifying such equations following from the characterization above. We will start in this paper with the simplest case namely when the Lagrangian is of the first order and the field is scalar i.e. $N = 1$.

In the case of first order Lagrangians one can use the formalism described in [5] which is very well suited for the study of Lagrangian systems with groups of symmetries. Applying this formalism we will be able to write down a rather general equation with universal invariance for a scalar field.

In Section 2 we present the general formalism for the case of a scalar field. In this case a rather complete discussion is possible. In Section 3 we derive the result announced above.

2 A Geometric Setting for the Lagrangian Formalism

2.1 The geometric setting of the Lagrangian theory in particle mechanics is usually based on the Poincaré-Cartan 1-form, but it is also possible to use a 2-form having as the associated system exactly the Euler-Lagrange equations [6]. This point of view was intensively exploited by Souriau [7] in connection with the Hamiltonian formalism. The proper generalization of these ideas to classical field theory is due to Krupka, Betounes and Rund [8]-[11]. We will follow the presentation from [5], but we will study, for simplicity, directly the case of a scalar field.

2.2 Let S be a differentiable manifold of dimension $n + 1$. The first order Lagrangian formalism is based on an auxiliary object, namely the bundle of 1-jets of n -dimensional submanifolds of S ,

$$J_n^1(S) \equiv \cup_{p \in S} J_n^1(S)_p$$

where $J_n^1(S)_p$ is the manifold of n -dimensional linear subspaces of the tangent space $T_p(S)$ at S in the point $p \in S$. This manifold is naturally fibered over S ; let us denote by π the canonical projection and construct a system of charts adapted to this fibered structure. We choose a system of local coordinates (x^μ, ψ) on the open set $U \subseteq S$; here $\mu = 1, \dots, n$. Then on the open set $V \subseteq \pi^{-1}(U)$ we shall choose the local coordinate system (x^μ, ψ, ψ_μ) defined as follows: if (x^μ, ψ) are the coordinates

of $p \in U$ then the n -dimensional hyperplane in $T_p(S)$ corresponding to (x^μ, ψ, ψ_μ) is spanned by the tangent vectors:

$$\frac{\delta}{\delta x^\mu} \equiv \frac{\partial}{\partial x^\mu} + \psi_\mu \frac{\partial}{\partial \psi}. \quad (1)$$

We will systematically use the summation convention over the dummy indices.

By an *evolution space* we mean any (open) subbundle E of $J_n^1(S)$. Note that $\dim(J_n^1(S)) = 2n + 1$.

2.3 Let us define for any evolution space E :

$$\Lambda_{LS} \equiv \{\sigma \in \wedge^{n+1}(J_n^1(S)) | i_{Z_1} i_{Z_2} \sigma = 0, \forall Z_1, Z_2 \in Vect(E) \text{ vertical}\}. \quad (2)$$

A vector field $Z \in Vect(E)$ is *vertical* if and only if $\pi_* Z = 0$. It is clear that any $\sigma \in \Lambda_{LS}$ can be written in the local coordinates from above as follows:

$$\sigma = \varepsilon_{\mu_1, \dots, \mu_n} (\sigma^{\mu_0} d\chi_{\mu_0} \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n} + n\sigma^{\mu_0\mu_1} d\chi_{\mu_0} \wedge \delta\psi \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_n}) \quad (3) \\ + n!\tau\delta\psi \wedge dx^1 \wedge \dots \wedge dx^n.$$

Here $\sigma^\mu, \sigma^{\mu_0\mu_1}$ and τ are smooth functions on E ,

$$\delta\psi \equiv d\psi - \psi_\mu dx^\mu \quad (4)$$

and $\varepsilon_{\mu_1, \dots, \mu_n}$ is the signature of the permutation $(1, \dots, n) \mapsto (\mu_1, \dots, \mu_n)$.

One can verify directly by performing a change of charts $(x^\mu, \psi) \mapsto (y^\mu, \zeta)$ inducing $(x^\mu, \psi, \psi_\mu) \mapsto (y^\mu, \zeta, \zeta_\mu)$ that the following equations have an intrinsic global meaning:

$$\sigma^\mu = 0 \quad (5)$$

$$\sigma^{\mu\nu} = \sigma^{\nu\mu}. \quad (6)$$

Any closed element $\sigma \in \Lambda_{LS}$ verifying (5) and (6) will be called a *Lagrange-Souriau form on E* (LS-form). Such a σ is of the form:

$$\sigma = n\varepsilon_{\mu_1, \dots, \mu_n} \sigma^{\mu_0\mu_1} d\psi_{\mu_0} \wedge \delta\psi \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_n} + n!\tau\delta\psi \wedge dx^1 \wedge \dots \wedge dx^n. \quad (7)$$

The closedness condition

$$d\sigma = 0 \quad (8)$$

gives explicetely:

$$\frac{\partial \sigma^{\mu\nu}}{\partial \psi_\rho} = \frac{\partial \sigma^{\mu\rho}}{\partial \psi_\nu} \quad (9)$$

$$\frac{\delta \sigma^{\mu\nu}}{\delta x^\nu} + \frac{\partial \tau}{\partial \psi_\mu} = 0. \quad (10)$$

We will call (6)+(9)+(10) the *structure equations*.

A *Lagrangian system over S* is a couple (E, σ) with $E \subseteq J_n^1(S)$ an evolution space over S and σ a Lagrange-Souriau form on E .

There is a natural *equivalence* relation between two such systems, (E_1, σ_1) and (E_2, σ_2) over the same manifold S i.e. one must have a map $\alpha \in Diff(S)$ such that $\dot{\alpha}(E_1) = E_2$ and:

$$(\dot{\alpha})^* \sigma_2 = \sigma_1. \quad (11)$$

where $\dot{\alpha} \in Diff(J_n^1(S))$ is the natural lift of α .

2.4 An *evolutions* is an immersions $\Psi : M \rightarrow S$, where M is some n -dimensional manifold (the "space-time" manifold).

Let (E, σ) be a Lagrangian system over S ; one says that $\Psi : M \rightarrow S$, verifies the *Euler-Lagrange equations* if

$$(\dot{\Psi})^* i_Z \sigma = 0 \quad (12)$$

for any vector field $Z \in Vect(E)$. Here $\dot{\Psi} : M \rightarrow J_n^1(S)$ is the natural lift of Ψ .

In local coordinates one can arrange such that Ψ has the form $x^\mu \mapsto (x^\mu, \Psi(x))$; then $\dot{\Psi} : M \rightarrow J_n^1(S)$ is given by $x^\mu \mapsto (x^\mu, \Psi(x), \frac{\partial \Psi}{\partial x^\mu}(x))$ and (12) have the local expression:

$$\sigma^{\mu\nu} \circ \dot{\Psi} \frac{\partial^2 \Psi}{\partial x^\mu \partial x^\nu} - \tau \circ \dot{\Psi} = 0. \quad (13)$$

An interesting result following directly from this equation is

Lemma The Euler-Lagrange equations are trivial *iff* $\sigma = 0$.

2.5 We come now to the notion of symmetry. By a *symmetry* of the Euler-Lagrange equations we understand a map $\phi \in Diff(S)$ such that if $\Psi : M \rightarrow S$ is a solution of these equations, then $\phi \circ \Psi$ is a solution of these equations also.

In the case of a scalar field one can completely describe the structure of a symmetry. We have:

Theorem 1: *Let (E, σ) be a Lagrangian system for a scalar field and $\phi \in Diff(S)$ a symmetry. Then there exists $\rho \in \mathcal{F}(E)$ such that*

$$(\dot{\phi})^* \sigma = \rho \sigma. \quad (14)$$

The function ρ must satisfy the equation

$$d\rho \wedge \sigma = 0 \quad (15)$$

or, in local coordinates:

$$\tau \frac{\partial \rho}{\partial \psi_\mu} + \sigma^{\mu\nu} \frac{\delta \rho}{\delta x^\nu} = 0 \quad (16)$$

$$\sigma^{\mu\nu} \frac{\partial \rho}{\partial \psi_\lambda} - \sigma^{\mu\lambda} \frac{\partial \rho}{\partial \psi_\nu} = 0. \quad (17)$$

Proof: Because Z in (12) is arbitrary, one easily discovers that ϕ is a symmetry *iff*

$$(\dot{\Psi})^* i_Z \sigma = 0 \implies (\dot{\Psi})^* (\dot{\phi})^* i_Z \sigma = 0, \quad \forall Z \in Vect(E), \quad \forall \Psi : M \mapsto S.$$

We denote for simplicity

$$\tilde{\sigma} \equiv (\dot{\phi})^* \sigma. \quad (18)$$

One can show immediately that $\tilde{\sigma}$ is a LS-form so it has the structure given by (7) with $\sigma^{\mu\nu} \mapsto \widetilde{\sigma^{\mu\nu}}$ and $\tau \mapsto \tilde{\tau}$. It follows from above that we have:

$$\sigma^{\mu\nu} \circ \dot{\Psi} \frac{\partial^2 \Psi}{\partial x^\mu \partial x^\nu} - \tau \circ \dot{\Psi} = 0 \implies \widetilde{\sigma^{\mu\nu}} \circ \dot{\Psi} \frac{\partial^2 \Psi}{\partial x^\mu \partial x^\nu} - \tilde{\tau} \circ \dot{\Psi} = 0 \quad (19)$$

(see (13).) Equivalently:

$$\sigma^{\mu\nu} \psi_{\{\mu\nu\}} - \tau = 0 \implies \widetilde{\sigma^{\mu\nu}} \psi_{\{\mu\nu\}} - \tilde{\tau} = 0 \quad (20)$$

where $\psi_{\{\mu\nu\}}$ is an arbitrary real symmetric matrix. In fact, it is more appropriate to consider expressions of the type appearing in (20) as functions on $J_n^2(S)$.

It is not hard to prove that (20) implies the existence of $\rho \in \mathcal{F}(E)$ such that:

$$\begin{aligned} \widetilde{\sigma^{\mu\nu}} \psi_{\{\mu\nu\}} - \tilde{\tau} &= \rho(\sigma^{\mu\nu} \psi_{\{\mu\nu\}} - \tau) \iff \\ \tilde{\tau} &= \rho\tau, \quad \widetilde{\sigma^{\mu\nu}} = \rho\sigma^{\mu\nu}. \end{aligned}$$

So, we find out that:

$$\tilde{\sigma} = \rho\sigma. \quad (21)$$

But, as noted above, $\tilde{\sigma}$ is a LS-form, so $\rho\sigma$ must be a LS-form. From the definition of a LS-form it is clear that only the closedness condition (15) is missing. The derivation of (16) and (17) is elementary. \square

Remarks:

1) If ρ is not locally constant, then (15) implies that σ is of the form

$$\sigma = d\rho \wedge \omega \quad (22)$$

with ω a n -form.

2) Let us suppose that the Lagrangian system (E, σ) is *non-degenerated*, that's it:

$$\det(\sigma^{\mu\nu}) \neq 0. \quad (23)$$

This condition has a global intrinsic meaning as it easily follows performing a change of charts. (The condition of non-degeneracy ensures that the Euler-Lagrange equations (13) can be "solved" with respect to the second order derivatives and the Cauchy problem can be well defined.) If we have (23) then one finds from (17) that $\frac{\partial \rho}{\partial \psi_\lambda} = 0$. Next, (16) gives $\frac{\partial \rho}{\partial \psi} = 0$ and $\frac{\partial \rho}{\partial x^\mu} = 0$ so ρ is locally constant. This result is a sort of Lee-Hwa Chung theorem (see e.g. [12]) for the Lagrangian formalism.

3) The case $\rho = 1$ corresponds to the so-called *Noetherian symmetries*. For a detailed discussion see [5]. \square

If a group G act on S : $G \ni g \mapsto \phi_g \in \text{Diff}(S)$ then we say that G is a *group of (Noetherian) symmetries* for (E, σ) if for any $g \in G$, ϕ_g is a (Noetherian) symmetry. In particular we have:

$$(\dot{\phi}_g)^* \sigma = \rho_g \sigma. \quad (24)$$

It is considered of physical interest to solve the following classification problem: given the manifold S with an action of some group G on S , find all Lagrangian systems (E, σ) where $E \subseteq J_n^1(S)$ is on open subset and G is a group of (Noetherian) symmetries for (E, σ) . This goal will be achieved by solving simultaneously (6), (9), (10), (16), (17) and (24) in local coordinates and then investigating the possibility of globalizing the result.

2.6 We close this section explaining the connection with the usual Lagrangian formalism. We can consider that the open set $V \subseteq \pi^{-1}(U)$ is simply connected by choosing it small enough.

From the structure equations (6), (9) and (10) one easily finds out that there exists a (local) function L on V such that:

$$\sigma^{\mu\nu} = \frac{\partial^2 L}{\partial \psi_\mu \partial \psi_\nu} \quad (25)$$

and

$$\tau = \frac{\partial L}{\partial \psi} - \frac{\delta}{\delta x^\mu} \left(\frac{\partial L}{\partial \psi_\mu} \right). \quad (26)$$

Now (13) takes the usual form for the Euler-Lagrange equations. L is called a *local Lagrangian*. If σ is given by (7) but with the coefficient functions as in (25) and (26) above, then we denote it by σ_L .

3 Universal Invariance

3.1 In the framework developped in Section 2, let us consider that $S = \mathbb{R}^n \times \mathbb{R}$ with global coordinates (x^μ, ψ) , ($\mu = 1, \dots, n$). We can take $E = J_n^1(S) \simeq \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ with global coordinates (x^μ, ψ, ψ_μ) . If $F \in \text{Diff}(\mathbb{R})$, let us consider $\phi_F \in \text{Diff}(S)$ given by:

$$\phi_F(x, \psi) = (x, F(\psi)). \quad (27)$$

By definition, the Lagrangian system (E, σ) defined above has *universal invariance* if ϕ_F is a symmetry, i.e. (see (24)) we have:

$$(\dot{\phi}_F)^* \sigma = \rho_F \sigma \quad (28)$$

for some functions $\rho_F \in \mathcal{F}(E)$ verifying (16) and (17). It is easy to show that ρ_F must verify the following consistency relation:

$$\rho_{F_1} \circ \dot{\phi}_{F_2} \rho_{F_2} = \rho_{F_1 \circ F_2} \quad (29)$$

which is easily recognized as a cocycle condition. One would be tempted to try to solve this cohomology problem. This can be done under some reasonable smoothness conditions, but we will prefer to circumvent this analysis.

3.2 We substitute (7) into (28) and get equivalently:

$$(F')^2 \sigma^{\mu\nu} \circ \dot{\phi}_F = \rho_F \sigma^{\mu\nu} \quad (30)$$

$$F'(\tau \circ \dot{\phi}_F - F'' \sigma^{\mu\nu} \circ \dot{\phi}_F \psi_\mu \psi_\nu) = \rho_F \tau. \quad (31)$$

From (30) it easily follows that ρ_F is a coboundary i.e. is of the form

$$\rho_F = b \circ \dot{\phi}_F b^{-1} \quad (32)$$

for some function $b \in \mathcal{F}(E)$. In fact, one can show that the most general solution of (29) is of this type.

3.3 We will study in fact only a particular case which covers the equations from [1]-[4], namely when b is a homogeneous function of degree $p \in \mathbb{N}$ in the variables ψ_μ . Then from (32) we have:

$$\rho_F = (F')^p. \quad (33)$$

We insert (33) into (30)+(31) and consider that F is an infinitesimal diffeomorphism i.e.

$$F(\psi) = \psi + \theta(\psi) \quad (34)$$

with θ infinitesimal but otherwise arbitrary. We obtain:

$$\frac{\partial \sigma^{\mu\nu}}{\partial \psi} = 0 \quad (35)$$

$$\psi_\lambda \frac{\partial \sigma^{\mu\nu}}{\partial \psi_\lambda} = (p-2) \sigma^{\mu\nu} \quad (36)$$

$$\frac{\partial \tau}{\partial \psi} = 0 \quad (37)$$

$$\psi_\lambda \frac{\partial \tau}{\partial \psi_\lambda} = (p-1) \tau \quad (38)$$

$$\sigma^{\mu\nu} \psi_\mu \psi_\nu = 0. \quad (39)$$

From the consistency equation (16) we obtain

$$\sigma^{\mu\nu} \psi_\mu = 0 \quad (40)$$

so (39) is redundant. Equation (17) is identically satisfied.

3.4 We analyse now the system (35)-(38)+(40). First, we concentrate on the functions $\sigma^{\mu\nu}$. Let us note that (36) is the infinitesimal form of the homogeneity property:

$$\sigma^{\mu\nu}(x, \lambda \psi_\mu) = \lambda^{p-2} \sigma^{\mu\nu}(x, \psi_\mu), \forall \lambda \in \mathbb{R}^*. \quad (41)$$

In the chart $\psi_0 \neq 0$ this means that $\sigma^{\mu\nu}$ is of the following form:

$$\sigma^{\mu\nu} = \psi_0^{p-2} s^{\mu\nu} \circ \pi \quad (42)$$

where $s^{\mu\nu}$ is a smooth function of the variables x, y_1, \dots, y_{n-1} and we have defined

$$\pi(x, \psi_1, \dots, \psi_n) = \left(x, \frac{\psi_1}{\psi_0}, \dots, \frac{\psi_{n-1}}{\psi_0} \right). \quad (43)$$

From (6) we have

$$s^{\mu\nu} = s^{\nu\mu} \quad (44)$$

and from (40):

$$s^{00} = \sum_{i,j=1}^{n-1} y_i y_j s^{ij} \quad (45)$$

$$s^{0i} = - \sum_{j=1}^{n-1} y_j s^{ij}. \quad (46)$$

We still have at our disposal the structure equation (9). It is convenient to define the operator

$$D \equiv \sum_{j=1}^{n-1} y_j \frac{\partial}{\partial y_j}. \quad (47)$$

Then (9) is equivalent to:

$$\frac{\partial s^{00}}{\partial y_i} = (p - 2 - D) s^{0i} \quad (48)$$

$$\frac{\partial s^{0i}}{\partial y_i} = (p - 2 - D) s^{ij} \quad (49)$$

$$\frac{\partial s^{ij}}{\partial y_k} = \frac{\partial s^{ik}}{\partial y_j} \quad (50)$$

If we insert (46) into (49) we obtain:

$$(p - 1) s^{ij} = 0. \quad (51)$$

Analogously, if we insert (45) and (46) into (48) we get:

$$(p - 1) \sum_{j=1}^{n-1} y_j s^{ij} = 0. \quad (52)$$

If $p \neq 1$ then it easily follows that

$$\sigma = 0 \quad (53)$$

so we are left only with the case $p = 1$. In this case (51) and (52) are becoming identities and s^{ij} are constrained only by (44) and (50). It follows that there exists a function l depending on x and y_1, \dots, y_{n-1} such that

$$s^{ij} = \frac{\partial^2 l}{\partial y_i \partial y_j}. \quad (54)$$

Then we get from (45) and (46):

$$s^{i0} = -D \frac{\partial l}{\partial y_i} \quad (55)$$

and

$$s^{00} = (D^2 - D)l. \quad (56)$$

The structure of the functions $s^{\mu\nu}$ is completely elucidated. If we define:

$$L_0 \equiv \psi_0 l \circ \pi \quad (57)$$

then it is elementary to prove that we have (25) with $L \rightarrow L_0$.

If we define

$$\sigma' = \sigma - \sigma_{L_0} \quad (58)$$

then it is easy to analyse the structure of this auxilliary LS-form which also verifies the invariance condition (28)+(33).

The final result can be summarized as follows:

Theorem 2: *Let (E, σ) be a first-order Lagrangian system for a scalar field having universal invariance (28) with ρ_F given by (33). Then we have non-trivial solutions only for $p = 1$. In this case we have*

$$\sigma = \sigma_L \quad (59)$$

with

$$L = \psi_0 l \circ \pi + \psi \tau. \quad (60)$$

Here l is a smooth function depending on x and y_1, \dots, y_{n-1} and τ is only x -dependent. The corresponding Euler-Lagrange equations (13) are, in the notations: $\Psi_\mu \equiv \frac{\partial \Psi}{\partial x^\mu}$, $\Psi_{\{\mu\nu\}} \equiv \frac{\partial^2 \Psi}{\partial x^\mu \partial x^\nu}$

$$\sum_{i,j=1}^{n-1} \frac{\partial^2 l}{\partial y_i \partial y_j} \left(x, \frac{\Psi_i}{\Psi_0} \right) \Psi_0^{-3} (\Psi_i \Psi_j \Psi_{\{00\}} - \Psi_0 \Psi_i \Psi_{\{0j\}} - \Psi_0 \Psi_j \Psi_{\{0i\}} + \Psi_0^2 \Psi_{\{ij\}}) - \tau = 0. \quad (61)$$

Remarks:

4) For $n = 2$ we get

$$\frac{\partial^2 l}{\partial y^2} \left(x, \frac{\Psi_1}{\Psi_0} \right) \Psi_0^{-3} (\Psi_1^2 \Psi_{\{00\}} - 2\Psi_0 \Psi_1 \Psi_{\{01\}} + \Psi_0^2 \Psi_{\{11\}}) - \tau = 0. \quad (62)$$

If we take $\tau = 0$ we get $(\dots) = 0$ which is the equation appearing in [4].

5) Because $p = 1$, the universal invariance is not a Noetherian symmetry. \square

4 Final Comments

There are a number of results obtained in this paper which will interesting to generalize.

First, one could try to extend the analysis above to the case of a field with more than one component. This extension seems possible and plausible, but there might appear some technical problems.

Next, we come to the universal invariance. Can one study the general case (32)? This seems to be a rather complicated problem.

Finally, one should like to generalize these results to higher-order Lagrangian systems. This problem is more manageable and some results in this direction will be reported soon. We will have to use a completely different method, because a generalization of the formalism from section 2 to higher-order Lagrangian systems is not available.

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